TRINITY COLLEGE

Semester Two Examination, 2021

Question/Answer booklet

MATHEMATICS METHODS UNITS 3&4

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words ______________________________

____

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(5 marks)**

A summary of the lengths of a large sample of nails from a production line are shown below.

Length, L mm	Relative frequency
$147 < L \le 148$	0.17
$148 < L \le 149$	0.13
$149 < L \le 150$	0.21
$150 < L \le 151$	0.19
$151 < L \le 152$	0.16
$152 < L \le 153$	0.14

(a) What proportion of nails are longer than 149 mm? (1 mark)

Solution
$p = 1 - 0.13 - 0.17 = 0.7$
Specific behaviours
✓ correct proportion

(b) Determine the probability that a randomly selected nail from the production line is longer than 150 mm given that it is no longer than 152 mm. (2 marks)

Solution
$P(L > 150 \mid L \le 152) = \dfrac{0.19 + 0.16}{1 - 0.14} = \dfrac{35}{86}$
Specific behaviours
✓ indicates use of correct relative frequencies ✓ simplifies to proper fraction

(c) State, with reasons, whether the data suggests that the nail lengths are normally distributed. (2 marks)

Solution
Not normally distributed. The relative frequencies do not reflect the bell shaped outline of a normal distribution and appear closer to a uniform distribution.
Specific behaviours
✓ states no ✓ justifies response

Question 2 **(5 marks)**

(a) Determine $\int \frac{2x+2}{x^2+2x-3}\,dx,\ x>1.$ (2 marks)

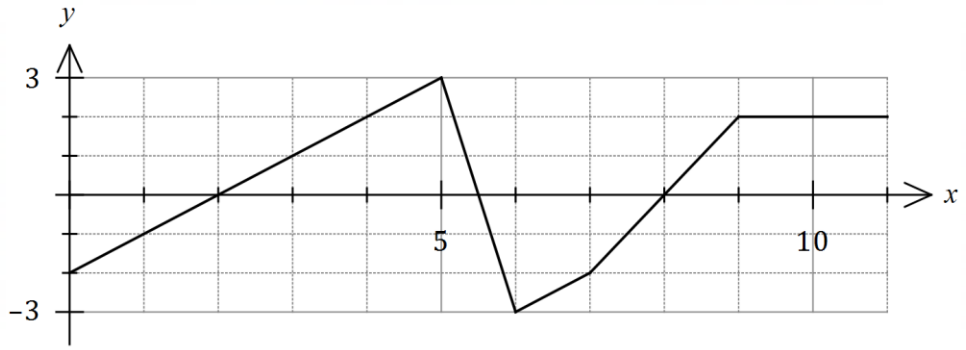
Solution
$\int \frac{2x+2}{x^2+2x-3}\,dx = \ln(x^2+2x-3)+c$
Specific behaviours
✓ antiderivative ✓ includes constant of integration

(b) The line $y = 10 - 2x$ intersects the curve $y = \frac{8}{x}$ at $(1, 8)$ and $(4, 2)$. Determine the area trapped between line and the curve. (3 marks)

Solution
$A = \int_1^4 10 - 2x - \frac{8}{x}\,dx$ $= [10x - x^2 - 8\ln x]_1^4$ $= [40 - 16 - 8\ln 4] - [10 - 1 - 0]$ $= 15 - 8\ln 4\ \ (= 15 - 16\ln 2)$ sq units
Specific behaviours
✓ writes correct integral ✓ antidifferentiates correctly ✓ substitutes and simplifies

Question 3 **(7 marks)**

The graph of $y = f(x)$ consists of line segments, as shown below.



Evaluate each of the following:

(a) $\displaystyle\int_3^5 f(x)\,dx.$ (1 mark)

Solution
$\int = 4$
Specific behaviours
✓ correct value

(b) $\displaystyle\int_0^5 f(x)\,dx.$ (2 marks)

Solution
$\int_0^2 + \int_2^5 = -2 + 4.5 = 2.5$
Specific behaviours
✓ indicates use of signed area ✓ correct value

(c) $\displaystyle\int_2^7 2f(x)\,dx.$ (2 marks)

Solution
$2\left(\int_2^5 + \int_5^6 + \int_6^7\right) = 2(4.5 + 0 - 2.5) = 2 \times 2 = 4$
Specific behaviours
✓ indicates use of linearity ✓ correct value

(d) $\displaystyle\int_5^9 (f(x) + 1)\,dx.$ (2 marks)

Solution
$\int_5^6 f + \int_6^7 f + \int_7^9 f + \int_5^9 1\,dx = 0 - 2.5 + 0 + 4 = 1.5$
Specific behaviours
✓ indicates use of additivity ✓ correct value

Question 4 **(7 marks)**

The curve $y = 4x + \dfrac{2}{x^2}$ has one stationary point.

(a) Obtain expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. (2 marks)

Solution
$\dfrac{dy}{dx} = 4 - \dfrac{4}{x^3}$ $\dfrac{d^2y}{dx^2} = \dfrac{12}{x^4}$
Specific behaviours
✓ first derivative   ✓ second derivative

(b) Determine the coordinates of the stationary point and determine its nature. (4 marks)

Solution
$\dfrac{dy}{dx} = 0 \Rightarrow 4 = \dfrac{4}{x^3} \Rightarrow x = 1$ $y = 4 + 2 = 6$ Stationary point at $(1,6)$. $x = 1 \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{12}{x^4} = 12$ Hence stationary point is a minimum.
Specific behaviours
✓ equates first derivative to zero and solves   ✓ calculates coordinates   ✓ evaluates second derivative at point   ✓ states minimum

(c) Explain why the curve has no point of inflection. (1 mark)

Solution
There is no value of x for which $\frac{d^2y}{dx^2} = 0$.
Specific behaviours
✓ explains using second derivative

Question 5 **(7 marks)**

(a) Let $F(x) = \int_0^x \cos 3\theta \, d\theta$.

Express $F(x)$ as a function of x and hence evaluate $F\left(\frac{\pi}{4}\right)$.

(3 marks)

Solution
$F(x) = \left[\frac{1}{3}\sin 3\theta\right]_0^x$ $= \frac{1}{3}\sin 3x$ $F\left(\frac{\pi}{4}\right) = \frac{1}{3}\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{6}$
Specific behaviours
✓ correct antiderivative   ✓ correct function   ✓ evaluates

(b) Let $g(x) = \dfrac{e^{1-x}}{1-x}$.

(i) Show that $g'(x) = \dfrac{x\, e^{1-x}}{(1-x)^2}$. (2 marks)

Solution
$g'(x) = \dfrac{(-e^{1-x})(1-x) - (e^{1-x})(-1)}{(1-x)^2}$ $= \dfrac{x\, e^{1-x}}{(1-x)^2}$
Specific behaviours
✓ shows correct u' and v' ✓ shows correct structure of quotient rule

(ii) Hence, or otherwise, evaluate $\displaystyle\int_{-1}^{0} \frac{2x\, e^{1-x}}{(1-x)^2} dx$. (2 marks)

Solution
$2\displaystyle\int_{-1}^{0} \frac{x\, e^{1-x}}{(1-x)^2} dx = 2\left[\frac{e^{1-x}}{1-x}\right]_{-1}^{0}$ $= 2e - e^2$
Specific behaviours
✓ indicates correct antiderivative   ✓ evaluates

Question 6 **(7 marks)**

(a) By first using log laws, or otherwise, determine $\frac{d}{dx}\left(\ln\left(e^{2x}\sqrt{x^3+1}\right)\right)$ in simplest form. (3 marks)

Solution
$\ln\left(e^{2x}\sqrt{x^3+1}\right) = \ln e^{2x} + \ln\left(\sqrt{x^3+1}\right)$ $= 2x + \frac{1}{2}\ln(x^3+1)$ $\frac{d}{dx}\left(2x + \frac{1}{2}\ln(x^3+1)\right) = 2 + \frac{3x^2}{2(x^3+1)}$ $= \frac{4x^3+3x^2+4}{2(x^3+1)}$ $= \frac{4x^3+3x^2+4}{2x^3+2}$
Specific behaviours
✓ uses one log law appropriately   ✓ uses second log law appropriately   ✓ correctly differentiates (and simplifies to any of three forms shown)

(b) The function $f(x) = x^2\ln\left(\frac{x}{2}\right)$ for $x > 0$ has one stationary point, a global minimum.

Determine the minimum value of the function. (4 marks)

Solution
$f'(x) = 2x\ln\left(\frac{x}{2}\right) + x^2\left(\frac{1}{x}\right)$ $= 2x\ln\left(\frac{x}{2}\right) + x$ $= x\left(2\ln\left(\frac{x}{2}\right) + 1\right)$ Stationary when: $f'(x) = 0 \Rightarrow \ln\left(\frac{x}{2}\right) = -\frac{1}{2}$ $\frac{x}{2} = e^{-\frac{1}{2}}$ $x = 2e^{-\frac{1}{2}}$ Minimum value: $f\left(2e^{-\frac{1}{2}}\right) = 4e^{-1}\ln\left(e^{-\frac{1}{2}}\right)$ $= \frac{4}{e} \times -\frac{1}{2} = -\frac{2}{e}$
Specific behaviours
✓ uses product rule correctly   ✓ obtains derivative   ✓ obtains root of derivative   ✓ calculates minimum value

Question 7 **(8 marks)**

The random variable X is defined by $P(X = x) = \begin{cases} k\log_3(x+2) & x = 1, 25, 79 \\ 0 & \text{elsewhere} \end{cases}$

(a) Determine the value of the constant k. (2 marks)

Solution
$k(\log_3 3 + \log_3 27 + \log_3 81) = 1$ $k(1 + 3 + 4) = 1$ $k = \frac{1}{8}$
Specific behaviours
✓ equation for k ✓ correct value

(b) Calculate the expected value of X. (2 marks)

Solution
$E(X) = 1 \times \frac{1}{8} + 25 \times \frac{3}{8} + 79 \times \frac{1}{2}$ $= \frac{76}{8} + \frac{79}{2} = 9.5 + 39.5 = 49$
Specific behaviours
✓ indicates $\sum xp$ ✓ correct $E(X)$

The Bernoulli random variable Y is solely dependent on X, so that $Y = 1$ when $X = 1$, and $Y = 0$ for all other values of X.

(c) Determine

(i) $P(Y = 0)$. (1 mark)

Solution
$P(Y = 0) = 1 - P(X = 1) = \frac{7}{8}$
Specific behaviours
✓ correct probability

(ii) $E(Y)$. (1 mark)

Solution
$E(Y) = 0 \times \frac{7}{8} + 1 \times \frac{1}{8} = \frac{1}{8}$
Specific behaviours
✓ correct value

(iii) $\text{Var}(3Y + 1)$. (2 marks)

Solution
$\text{Var}(Y) = \frac{7}{8} \times \frac{1}{8} = \frac{7}{64}$ $\text{Var}(3Y + 1) = 3^2 \times \frac{7}{64} = \frac{63}{64}$
Specific behaviours
✓ $\text{Var}(Y)$ ✓ $\text{Var}(3Y + 1)$

Question 8 **(6 marks)**

In triangle ABC, the length a of the side opposite angle A is given by $a = \sqrt{11 - 4\cos A}$ cm.

Use the increments formula to calculate the approximate change in length of a as the size of angle A decreases from $\frac{15\pi}{45}$ to $\frac{14\pi}{45}$.

Solution
$a = (11 - 4\cos A)^{\frac{1}{2}}$ $\frac{da}{dA} = \frac{1}{2}(11 - 4\cos A)^{-\frac{1}{2}}(4\sin A)$ $= \frac{2\sin A}{\sqrt{11 - 4\cos A}}$ When $\angle A = \frac{\pi}{3}$: $\frac{da}{dA} = \frac{2\sin\frac{\pi}{3}}{\sqrt{11 - 4\cos\frac{\pi}{3}}} = 2\frac{\sqrt{3}}{2} \div \sqrt{9} = \frac{\sqrt{3}}{3}$ $\delta A = -\frac{\pi}{45}$ $\delta a \approx \frac{da}{dA} \times \delta A$ $\approx \frac{\sqrt{3}}{3} \times -\frac{\pi}{45}$ $\approx -\frac{\sqrt{3}\pi}{135}$ cm   Hence length decreases by approximately $\frac{\sqrt{3}\pi}{135}$ cm.
Specific behaviours
✓ indicates use of chain rule   ✓ correct derivative   ✓ evaluates derivative at initial angle   ✓ indicates incremental change   ✓ uses increments formula   ✓ states decrease in length with units